$$y_2(x,t)=Z[(D-C)^3+(D-B)^3],\ at^2/2<x<ct \tag{10b}$$

$$y_2(x,t)=Z[(D-B)^3-(D-A)^3],\ 0<x<at^2/2 \tag{10c}$$

in which

$$A(x,t)=(c^2/2a+ct+x)^{1/2}, B(x,t)=(c^2/2a+ct-x)^{1/2}$$

$$C(x,t)=(c^2/2a-ct+x)^{1/2}, D=(c^2/2a)^{1/2}$$

$$Y=Mg/[(2a)^{1/2}\rho c], Z=Mg(2a)^{1/2}/3cT \tag{11}$$

The kink angle under the mass at $x=at^2/2$ is obtained by differentiating the deflection formulas and calculating the slope differences. For the loading terms given by Eqs. (4 and 5), the kink angle Δ is

$$\Delta=\frac{Mg}{T}\left[\frac{1+(V/c)^2}{1-(V/c)^2}\right] \tag{12}$$

For the special case when $V=c$ and at the mass location

$$y_1=0.293\ Mg/\rho a,\quad y_2=0.155\ Mg/\rho a,\quad y=0.448\ Mg/\rho a \tag{13}$$

Numerical Results and Comparison with a Test

Deflection profiles of y_1 and y are shown in Fig. 1 for $M=100$ kg, $T=490$ kN, $c=307$ m/sec, $a=226$ m/sec^2, at $t=0.68$, 1.02 ane 1.29 sec. At these times the mass positions are $x=52$, 118, and 188 m. As shown in Fig. 1, maximum deflection is at the mass location.

To determine the adequacy of the simplified mathematical model used in the analyses, the cable deflection at a known position was monitored with a framing camera. A comparison of measured and predicted deflections at $x=122$ m is shown in Fig. 2. These data indicated that the predictions of cable motion are reasonably accurate.

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Effect of the Mass Center Shift for Force-Free Flexible Spacecraft

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Introduction

As spacecraft increase in size, the weight problem is becoming progressively critical. One way of reducing the weight is to make the structural members as light as possible. But reduced structural weight generally implies reduced stiffness, so that the flexibility becomes an important factor in the spacecraft dynamic characteristics. Indeed, flimsy elastic members undergo large elastic deformations which can have a significant effect on the spacecraft stability and control.

In describing the motion of a spinning spacecraft, it is often convenient to refer the motion to a system of axes with the origin at the spacecraft mass center C. For flexible spacecraft the mass center C generally shifts relative to the nominal undeformed position, so that the system kinetic energy contains terms involving the shifting of C.[1] Quite often these terms complicate the stability analysis appreciably. It is shown in Ref. 2, however, that for a force-free flexible spacecraft of a particular configuration the stability criteria are not affected adversely by ignoring the shifting of C. The question arises naturally as to whether this is true in general. It is the purpose of this paper to address this question and to generalize the results of Ref. 2. Indeed, the paper proves that for the general class of force-free single-spin flexible spacecraft it is possible to ignore the shifting of the mass center relative to the nominal undeformed state without affecting the stability criteria in any significant way. The errors in the stability criteria so derived not only tend to be very small, but they are also on the safe side because such criteria are conservative compared to those obtained by including the shifting of C. The simplification in the stability analysis achieved by ignoring the shifting of C fully justifies the relatively small loss in accuracy.

Stability Analysis

Consider a force-free single-spin flexible spacecraft. Assuming that the orbital motion is known (and can be ignored), it is shown[1] that the system kinetic energy for motion relative to C can be written in the form

$$T=\tfrac{1}{2}\omega^T J\omega+K^T\omega+\tfrac{1}{2}\int_m \dot{u}_c'^T\dot{u}_c' dm \tag{1}$$

where ω is the body angular velocity, J is inertia matrix in deformed state, $\boldsymbol{K}$ is the vector representing the angular momentum due to elastic velocities alone, and $\dot{u}_c'$ is the vector of these elastic velocities measured relative to a set of body axes with the origin at C. For a force-free system the potential energy V_{EL} is due entirely to elastic deformations. Its expression is not affected by the shifting of the mass center. Taking into account the angular momentum integral, we conclude from Ref. 1 that the system is asymptotically stable if the functional

$$\kappa=T_0+V_{EL} \tag{2}$$

is positive definite, where

$$T_0=\tfrac{1}{2}\beta^T J^{-1}\beta \tag{3}$$

in which β is the conserved angular momentum vector.

The term T_0 involves the shifting x_c, y_c, z_c of the mass center. Because x_c, y_c, and z_c involve integrals of the elastic deformations, they do not represent additional generalized coordinates. Their presence in T_0, however, complicates the stability analysis appreciably, so that an examination of their role in the analysis is of vital interest. To this end, let us observe that J can be written in the form

$$J=J_u-J_c \tag{4}$$

where J_u is the inertia matrix obtained by ignoring x_c, y_c, and z_c and

$$J_c=m\begin{bmatrix} y_c^2+z_c^2 & -x_cy_c & -x_cz_c \\ -x_cy_c & x_c^2+z_c^2 & -y_cz_c \\ -x_cz_c & -y_cz_c & x_c^2+y_c^2 \end{bmatrix} \tag{5}$$

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in which m is the total mass of the spacecraft. Whereas matrices J and J_u are positive definite, J_c is only positive. It follows that for any arbitrary vector α, the quadratic forms associated with J and J_u satisfy the inequality

$$\alpha^T J \alpha \leq \alpha^T J_u \alpha \quad (6)$$

Using the theorem of the next section, however, we conclude that

$$\beta^T J^{-1} \beta \geq \beta^T J_u^{-1} \beta \quad (7)$$

Hence, introducing the functional

$$\kappa_I = \tfrac{1}{2} \beta^T J_u^{-1} \beta + V_{EL} \quad (8)$$

and considering inequality, Eq. (7), we can write

$$\kappa \geq \kappa_I \quad (9)$$

so that the system is asymptotically stable if κ_I is positive definite.

The implication of inequality, Eq. (9), is that it is possible to use the testing functional κ_I for stability analysis instead of κ. Because κ_I is free of the terms involving the shifting of C, the stability analysis can be simplified considerably by using κ_I as a testing functional instead of κ. The conclusion is valid irrespective of the magnitude of x_c, y_c, and z_c. Of course, when x_c, y_c, and z_c are large the stability criteria derived by using κ_I as a testing functional instead of κ can be unduly restrictive. In most practical cases, however, x_c, y_c, and z_c are one order of magnitude smaller than the elastic displacements themselves, in which case no accuracy is sacrificed by using κ_I instead of κ.

A Theorem on Inequalities for Quadratic Forms

The simplification of the stability analysis resulting from the use of the testing functional κ_I instead of κ was based on the fact that if matrices J and J_u are such that inequality (6) is satisfied then matrices J^{-1} and J_u^{-1} satisfy inequality (7). Of course, matrices J and J_u do satisfy inequality (6), but it remains for us to prove that inequality (7) follows from inequality (6). Consider the following:

Theorem

Given two $n \times n$ matrices A and B which are symmetric and positive definite over the real number field R. If $x^T Ax \geq x^T Bx$ for any n-vector x over R, then $x^T A^{-1} x \leq x^T B^{-1} x$.

Proof:

Because A is symmetric and positive definite, there is an orthonormal matrix U such that[3]

$$A^{-1/2} = U \lambda^{-1/2} U^T \quad (10)$$

where λ is a diagonal matrix with its elements equal to the eigenvalues of the matrix A. The effect of the operation

$$C = A^{-1/2} B A^{-1/2} \quad (11)$$

is to transform the symmetric and positive definite matrix B into a matrix C which is also symmetric and positive definite, i.e.

$$C^T = (A^{-1/2} B A^{-1/2})^T = A^{-1/2} B^T A^{-1/2} = A^{-1/2} B A^{-1/2} = C \quad (12)$$

Similarly, there exists an orthonormal matrix V such that

$$V^T C V = V^T A^{-1/2} B A^{-1/2} V = \mu \quad (13)$$

where μ is a diagonal matrix.

Introducing the linear transformation

$$p = V^T A^{1/2} x \quad (14)$$

into the inequality $x^T Ax \geq x^T Bx$, where p is an n-vector, we obtain

$$p^T V^T A^{-1/2} A A^{-1/2} V p \geq p^T V^T A^{-1/2} B A^{-1/2} V p \quad (15)$$

which reduces to

$$p^T I p \geq p^T \mu p \quad (16)$$

where I is the identity matrix. Because A and B are positive definite, all the elements of the diagonal matrix μ are positive. It follows from inequality (16) that

$$p^T I^{-1} p \leq p^T \mu^{-1} p \quad (17)$$

Moreover, recalling inequalities (15) and (16), it follows that

$$p^T V^T A^{1/2} A^{-1} A^{1/2} V p \leq p^T V^T A^{1/2} B^{-1} A^{1/2} V p \quad (18)$$

Next, let

$$y = A^{1/2} V p = A^{1/2} V V^T A^{1/2} x = Ax \quad (19)$$

so that inequality (18) reduces to

$$y^T A^{-1} y \leq y^T B^{-1} y \quad (20)$$

Because A is symmetric and positive definite, we can show that A can be regarded as a linear transformation mapping the linear space into itself. This concludes the proof that $x^T A^{-1} x \leq x^T B^{-1} x$.

Conclusions

Considerable simplification of the stability analysis for flexible spacecraft can be achieved by ignoring the shifting of the spacecraft mass center relative to the nominal undeformed position. The resulting stability criteria are conservative compared with those obtained by including the shifting of the mass center, but in most practical cases the loss of accuracy is insignificant. To demonstrate the validity of the analysis, a new theorem on inequalities for quadratic forms is advanced and a proof of the theorem is provided.

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Nonequilibrium Nozzle Flow of a Nitrogen-Hydrogen Mixture

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THE high enthalpy nozzle flow of a nitrogen-hydrogen mixture is of interest from two points of view. First,

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